OPERATOR ALGEBRAS - THE FIRST FORTY YEARS

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1. Introduction. This article is intended to serve as a "road map" through the subject of Operator Algebras - for the purpose of putting the more detailed accounts that follow in understandable relation to one another and to the subject as a whole. Despite the title, this does not purport to be an historical treatment of the development of Operator Algebras - careful or otherwise. Insofar as historical statements are made, they are intended merely as "signposts" that may help the reader to understand why the subject developed in the way and order that it did. Even with these modest historical intentions, inevitably, there will be statements made that may be less than completely objective.

There are several excellent bibliographies of the subject available (see, for example, [8,88,91]). Those references that do appear in this article are only a sampler. They help to form the bibliographical framework on which this account is built. Many useful and important references that might well have appeared will not. Partly this is due to reasons of space and partly to the fact that they will not have come to mind at the appropriate moment. Much of this oversight will be rectified in the more detailed articles that follow.

2. A brief chronology. In a sense that is relatively rare in subjects based on important ideas, this subject has a discernible beginning. A paper published in 1930 by von Neumann [61] defines "rings of operators" (now called "von Neumann algebras") and proves his celebrated "Double Commutant Theorem".

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The collaboration between Murray and von Neumann that led to the fundamental series of papers on von Neumann algebras in general and the so-called, factors in particular began with a paper [58] that appeared in the 1936 Annals of Mathematics. Those von Neumann algebras whose centers consist of scalars, the factors, are subjected to an intensive general analysis through a theory that compares the "dimensions" of projections in the factors. A dimension function is defined on the lattice of projections in the factors. It is noted, on theoretical grounds, that there are several possible ranges for this dimension function. This range may be $[0,1,\ldots,n]$ with n a finite cardinal, or $[0,1,\ldots,\aleph_0]$. (The underlying Hilbert space is assumed, in this series of papers, to be separable.) In these cases, the factor is said to be of type I (respectively, I_n or I_∞). The range may be $[0,1]$ (after a suitable normalization) or $[0,\aleph_0]$. In these cases, the factor is said to be of type II (respectively, II_1 or II_∞). Finally the range may be $\{0,\aleph_0\}$, in which case the factor is said to be of type III. In this same paper, a class of examples of factors is constructed based on a countable (discrete) group acting by measure-preserving transformations on a measure space. By means of this class of examples, specific factors of all types were exhibited except for factors of types III. This construction has been extended with great profit and is of active current interest. We will have more to say about it shortly. The original construction is referred to as the "crossed product of an abelian von Neumann algebra by a discrete (countable) group" in present terminology. The concluding section of [58] introduces still another concept that forms the basis for an important current investigation - that of an algebra of unbounded operators. In earlier papers [64,65], von Neumann pointed out the subtleties involved in performing algebraic manipulations with unbounded operators. In [58] it is established that the operators (including unbounded operators) "affiliated" with a factor of type II_1 form an (associative) algebra.

The second paper of the series [59] appeared a year following the first. It deals almost exclusively with the trace function -

a linear extension to all operators of the dimension function on projections - on a factor of type II_1. What the extension of the dimension function must be is relatively clear from spectral theory. The linearity of this extension is a good deal less clear! In the process of establishing this linearity two techniques are developed that foreshadow later results of great importance. The first constructs a representation, in the simplified circumstances afforded by the trace, in a way that presages the GNS construction. The second represents the trace as a sum of vector states in a manner that illustrates the form of a normal state as a sum of vector states.

Between the appearances of the second and third papers in the "Rings of Operators" series, an important development took place in the subject. M. H. Stone identified (real) commutative norm-closed algebras of self-adjoint operators on a Hilbert space as the algebras of real-valued continuous functions on a compact Hausdorff space through their structure as real-ordered algebras [85]. This was a logical extension of his Boolean algebra representation theorem in which points in a topological space are extracted from a topological-algebraic structure.

The third paper [62] in the "Rings of Operators" series enlarged the construction of examples of factors in the first paper to permit groups of <u>measurability</u> (but not necessarily, <u>measure</u>)-preserving transformations. Through these examples, (especially when there was no non-trivial equivalent invariant measures), factors of type III were exhibited (and recognized) for the first time.

In the period between the appearance of the third and fourth papers of the series, several events occurred that were to have a profound effect on the subject. Near the beginning of the period, Segal [79] related harmonic analysis on locally compact topological groups to (non-commutative) self-adjoint operator algebras acting on a Hilbert space by means of an infinite-dimensional analogue of the complex group algebra construct. Throughout this interval, some of the basic tools of functional analysis were sharpened and developed (separation forms of the

Hahn-Banach theorem, the Alaoglu-Bourbaki theorem, the Krein-Millman theorem, the Krein-Smulyan theorem, etc.). At the end of the period, simultaneous with the publication of the fourth paper in the "Rings of Operators" series [60], the Gelfand-Neumark paper [15] appeared. In it, norm-closed self-adjoint subalgebras of $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on the Hilbert space $\mathcal{H}$, are characterized. In [15] Banach algebras with an adjoint (*-) operation satisfying certain properties (the so-called B*-algebras) are proved to be * isomorphic (and isometric) to norm-closed self-adjoint subalgebras of $\mathcal{B}(\mathcal{H})$. In the process of proving this, the work begun by Stone on topological-algebraic characterizations of C(X) is taken up; and C(X) is identified as a commutative B*-algebra.

The fourth paper in the "Rings of Operators" series [60] settles some of the major problems of the basic theory. Two such results are the classification of spatial action of factors in terms of algebraic structure and the existence of two factors of type II_1 that are not algebraically isomorphic. The spatial analysis is carried out by reduction to and uniqueness of the trace; so that factors of type III could not be included in this study. In the process of establishing that the classification of factors into types by the nature of the ranges of the dimension functions does not complete the algebraic classification of factors, Murray and von Neumann introduce a new technique of construction of factors of type II_1. For this they use (what, in more recent times, is called) the left regular representation of certain countably infinite groups and pass to the strong-operator closure of the algebra generated by the representing (unitary) operators. The groups in question are those for which each element distinct from the identity has an infinite number of conjugates. Some examples of these "i.c.c. groups" are the free non-abelian groups on two or more generators and the group of those permutations of the integers that move at most a finite number. The factors associated with each of the free groups is not isomorphic to the factor arising from the permutation group. This last factor is seen to be the strong-operator closure of an ascending sequence of finite type I factors. In the fourth paper of the series such factors are called "approximately finite" (and, more recently, "hyperfinite", "matricial"). In that paper, it is shown that all such factors of type II_1 are isomorphic.

During the seven years in which the four papers of the monumental "Rings of Operators" series were published and in the half-dozen years following their appearance they produced little visible effect on the rest of mathematics or other mathematicians. There is ample evidence that several of our most eminent mathematicians (then quite young) were intimately familiar with the details of these papers, and several others had attempted to master them. Many of the techniques so basic to the subject and so familiar to specialists today were undeveloped and even undiscovered at that time. To work in this area then meant pressing on with little chance of clearly significant advance on something that might just be a passing curiosity. The problems Murray and von Neumann had posed were clear cut but not easy to approach. The von Neumann algebra aspects of the subject languished for several years.

The Gelfand-Neumark paper [15] initiated the study of the norm-closed self-adjoint operator algebras, the C*-algebras. Its authors were certainly inspired by the "Rings of Operators" series - the last section of their paper proves interesting results on the ideal structure of factors. The techniques of their paper emphasize ideal structure, functionals and, to a lesser but vital extent, order structure in the Banach algebra. In [80] Segal fastens on the functional and order structure, isolating and developing a critical part of the Gelfand-Neumark argument to produce what is now known as the "GNS construction" - a method of associating with states (positive normalized linear functionals) of a C*-algebra an adjoint preserving (*-) representation of the algebra. Segal recognized the importance of the, then, recently-proved Krein-Millman theorem (in the theory of convexity in topological linear spaces) for this process - defining pure states (extreme points of the convex set of states) and showing that they correspond to irreducible representations of the algebra. He applied these results to the construction of (infinite-dimensional) irreducible unitary representations of locally compact groups. (See [16] as well.) The interest in Hilbert's Fifth Problem (are locally euclidean groups Lie groups?) and the possibility of solving it by producing, for locally compact groups, an analogue of the Peter-Weyl theory for compact groups was a major stimulus at this time. The search for a non-commutative Fourier-Plancherel-Weil transform and the general development of non-commutative harmonic analysis were important associated goals. Weil's profound book [101] had supplied a significant impetus to this program.

In this period, Segal published a paper [81] treating the foundations of quantum mechanics from the point of view of operator algebras. Von Neumann had certainly understood the importance of developing a theory of operator algebras as a rigorous framework for quantum physics. Segal's early work moved the focus to the area of C*-algebras - an important technical addition.

In the late 1940's Dixmier extended the concept of trace on a factor to that of a center-valued trace on a general von Neumann algebra [5]. Von Neumann's paper [63] expressing the general von Neumann algebra as a "generalized" direct sum (<u>direct integral</u>) of factors appeared in 1949 (in the form in which he had prepared the manuscript in 1938).

From 1946 to the beginning of the fifties an algebraic theory of von Neumann algebras developed [47,48,72]. Vital tools were fashioned for a systematic study of these algebras. The early 1950's saw a crucial shift of emphasis from the multiplicative and ideal structure of C*-algebras to their order structure [28,30]. This latter emphasis has dominated the technical development of the subject from that time to the present. The first half of the fifties saw, too, the study of general structure for the algebraic and spatial theory of C*-algebras [30,34], the beginning of a detailed structure theory for an important class of C*-algebras (the type I C*-algebras) [49] and the introduction, in algebraic form, of some of the important constructs from algebra - tensor products and crossed products [55,99,100].

This same period saw the emergence and elaboration of a viewpoint that is certainly the most basic background motivation in the subject of operator algebras - the C*-algebra as a "non-commutative" C(X) and the von Neumann algebra as a "non-commutative" measure algebra (of bounded Borel measurable functions on a measure space) [29,82] - though, of course, the non-commutative measure and integration theory is all but explicit in the third and fourth papers [62,60] of the "Rings of Operators" series. This view of the subject makes clear the fundamental role it must play as the framework for non-commutative real - analysis - and to the extent that real analysis is tied to geometry, particularly topology and differential geometry, as the basis for a development of non-commutative geometry. Some small elaboration of this last statement is necessary. In dealing with C*-algebras, an operator and its adjoint are treated on an equal footing. In the commutative case, the adjoint operator becomes the complex conjugate - so that, while we may be speaking of complex-valued functions in C(X), the real-valued functions are determining. For non-commutative complex function theory (holomorphic function theory) we must turn to the study of certain classes of non-self-adjoint operator algebras. While that subject is not within the scope of this article, it is worth noting that a healthy beginning has been made there [44,73].

In the early fifties, a parallel (but less active) development was taking place in the von Neumann algebra aspects of the subject. The Kaplansky Density theorem appeared [51]; a determinant theory in factors and its applications to the beginning of the study of infinite classical groups was developed [14,31,32,33]; an outer automorphism is constructed in a factor of type II_1 [8: Exercise 15, p. 288]; derivations of type I factors are shown to be inner [50]; the theory of normal states and their vector representations is completed [6,10,22,56]. This last development had the important consequence of freeing much of the study of von Neumann algebras from its reliance on the trace.

and, thereby, including the type III factors within its scope. In particular, the factors of type III are included, by these means, in the reduction of the spatial classification to the algebraic classification [22,56].

During this period the Hilbert Fifth Problem was solved affirmatively [17,57,103] by more or less topological and group structural techniques alone. One of the strong initial motivations for the study of C*-algebras and infinite-dimensional unitary representations of locally compact groups was removed by this. Nonetheless, the importance of this study for a theory of non-commutative harmonic analysis had become clear by this time [16,21,25,53,54,79]; and the group-operator algebra investigations continued. The "type problem" for locally compact groups came under special scrutiny - what are the possible types of the von Neumann algebras generated by the unitary representations of various classes of groups [7,25,45,52,54,95].

The initial period, during which Murray and von Neumann completed their fundamental work on factors, may be described as the strong, daring and brilliant beginning. The work of the later forties was innovative - but tentative and experimental. It was still not clear at that point whether these results were to be the basis for a subject or to become arcane curiosities. One of the (non-technical) aspects of the first half of the fifties was its answer to this question. Operator algebras was a subject with practitioners devoted to its development.

The mid-fifties saw the (general) solution of the unitary invariants problem for representations of C*-algebras [34,89]; the introduction of completely positive linear mappings [84] (the positive linear mappings had appeared in [30]); the abstract development [97] of the von Neumann diagonalization process [62] and the Dixmier process [5]; the construction of an automorphism (of a II_∞ factor with II_1 commutant) that scales the trace and is not unitarily implementable [35]; the group-measure space construct parallel to the approximately finite factors [11]; the simplification of some of the harder arguments of the "Rings of isomorphic factors of type III [71]; and the representation-independent characterizations of von Neumann algebras [74,38] (the first as a dual Banach space and the second as a monotone complete C*-algebra with "normal" states). This same period saw the understanding and sharpening of the techniques of the second dual (and the "universal representation") of a C*-algebra [83,90] for translating certain problems about such algebras into problems about von Neumann algebras (where they are often easily settled) and the affirmative solution to the problem of spectral synthesis in C*-algebras through a deeper understanding of pure states and irreducible representations [39].

In the mid to late fifties, some of the earlier technical problems that had concerned us were settled. Sakai [75] clarified the situation of the type of the tensor product (showing that tensoring by a type III algebra leads to type III von Neumann algebras). Takesaki [92] refined our techniques with general (norm-continuous) functionals on C*-algebras, making incisive use of the universal representation technique. Sakai [76] developed the important polar decomposition for normal functionals on von Neumann algebras. The problem of whether pure states have unique extensions from maximal abelian subalgebras of the algebra of all bounded operators on a Hilbert space to that algebra was settled negatively (and replaced by that problem for the "discrete" maximal abelian algebra and non-normal pure states of it) - this problem had been posed by the physical interpretation of the self-adjoint operators on a Hilbert space as the observables of an irreducible quantum mechanical system with finitely many degrees of freedom [46]. Sakai proved [50,77] Kaplansky's conjecture that a derivation of a C*-algebra into itself is (automatically) bounded.

The early sixties bears the stamp of Glimm's splendid work. His thesis [18] introduced and subjected to a penetrating analysis one of the most important classes of simple C*-algebras, the uniformly hyperfinite (uhf) C*-algebras. In [19], Glimm proved a non-commutative version of the Stone-Weierstrass theorem. Mackey's conjecture that the type I C*-algebras (groups) are precisely the ones with smooth duals is proved in [20]. (Dixmier [9] made valuable contributions to this success.) From the early to mid-sixties deeper ties between the obviously related subjects of operator algebras and quantum physics were established [1,24, 41,102]. Effros established the intimate connection among order, algebraic, and facial structure on the cone of positive elements, in a C*-algebra [12]. Takesaki answered important questions about the possible cross norms on tensor products of operator algebras [93].

In the mid sixties, derivations and automorphisms reigned. It had been assumed by all of us that, as with automorphisms (the "exponentials" of derivations), non-inner derivations of von Neumann algebras abound. The reverse turned out to be the case, and the proof that each such derivation is inner [26,40,42,78] led to a flurry of activity on automorphisms and derivations. In [43], automorphisms not on the surface of the ball of radius 2 with the identity automorphism as center were shown to be inner, and the structure of the automorphism group of a special class of C*-algebras was studied. Borchers [3] extended a result of [40] to groups of automorphisms of a von Neumann algebra in the context of the operator algebra formulation of quantum field theory. (See also Dell'Antonio's paper [4].) In this period, Pedersen introduced his important ideal in the framework of a non-commutative version of measure theory in C*-algebras [66,67,68,69], and Effros, Størmer and Topping [13,86,87,98] advanced our understanding of the Jordan structure of the family of self-adjoint operators in an operator algebra. (Compare [27] where this structure is first introduced as a mathematical model for quantum mechanics, [81] where it is directly related to the norm-closed algebras for the same purpose, and [29,30] where the first results are proved relating the Jordan structure to that of its enveloping C*-algebra.)

In 1967, two results appeared that were to affect the course of operator algebras profoundly. One, of Powers [70], established that there is a continuum of non-isomorphic, matricial factors of type III, the other, of Tomita [96] supplied, for a factor with a separating and generating vector, a fundamental construction relating the factor to its commutant and producing a quite unexpected one-parameter group of automorphisms of the factor.

As luck would have it, both results were discovered during the planning stage of the conference in Baton Rouge, Louisiana (the forerunner of the present conference). Both results were discussed at the conference and received the widest dissemination. At this same time Haag, Hugenholtz, and Winnink, in an important paper [23] relating quantum statistical mechanics and operator algebras introduced the KMS condition. The conjunction of KMS theory and that of Tomita's "modular" theory at the same conference could not have been more fortuitous, for these theories could hardly be more intimately related. In Takesaki's hands [94], the combined KMS and modular theory has become the driving force of the modern theory of operator algebras. Understanding the significance of the results of Powers and Tomita occupied the last part of the sixties. In particular, the classification of type III factors that appear as the (infinite) tensor product of finite type I factors by Araki and Woods [2] is one consequence of this process.

The explosion of results that has characterized the seventies is very much the result of the early groundwork and the remarkable breakthroughs of the mid to late sixties. It is, too, very much the subject of these volumes. In the pages that follow, a detailed account will be found of operator algebras as it is practiced today. We commend this exposition to the reader with the hope that our summary of the beginnings and middle period in the development of the theory of operator algebras helps in understanding the motivation for and significance of the results discussed.

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